Multi-hadron systems in LQCD

William Detmold
The College of William and Mary &
Thomas Jefferson National Accelerator Facility

Hadrons, Lattice QCD and Chiral Perturbation Theory, Graz, Sep 13th 2010
• $n<13$ pion or kaon systems
• 2- and 3- body interactions
• Meson condensates and screening of the static quark potential
• $n<4$ baryon systems
• Recent developments
  • Really large systems: $^{208}\pi^+$
  • Mixed species systems
Motivation

• Multi hadron interactions define nuclear physics

• Meson condensates: interesting state of matter with a complex phase diagram
  • finite $\mu_I$: BEC-BCS crossover
  • Spontaneous rotational symmetry breaking for $\mu_I \geq m_\rho$

• Kaon condensation may be phenomenologically relevant in n-stars [Kaplan/Nelson]

• Complexity frontier: precursor to nuclear systems
Few meson systems

Hadron scattering

- Maiani-Testa: extracting multi-hadron S-matrix elements from Euclidean lattice calculations of Green functions is impossible
- Lüscher: volume dependence of two-particle energy levels \( \Rightarrow \) scattering phase-shift up to inelastic threshold

\[
\Delta E(n) = \sqrt{|q(n)|^2 + m_A^2} + \sqrt{|q(n)|^2 + m_B^2} - m_A - m_B
\]

\[
q(n) \cot \delta(q(n)) = \frac{1}{\pi L} S \left( \frac{q(n)L}{2\pi} \right)
\]

\[
S(\eta) = \lim_{\Lambda \to \infty} \left[ \sum_{|\vec{n}| < \Lambda} \frac{1}{|\vec{n}|^2 - \eta^2 - 4\pi \Lambda} \right]
\]
Hadron scattering

- Maiani-Testa: extracting multi-hadron S-matrix elements from Euclidean lattice calculations of Green functions is impossible
- Lüscher: volume dependence of two-particle energy levels ⇒ scattering phase-shift up to inelastic threshold
- Exact relation provided $r \ll L$
- Used for $\pi \pi$, KK, NN, $\Lambda N$, ...
- What about $n>2$ hadrons?
Bosons in a box

- Large volume expansion of ground state energy of \( n \) meson system to \( 1/L^7 \)
- 2 & 3 body interactions (\( N \) body: \( L^{-3(N-1)} \))
- \( n=2 \): reproduces expansion of Lüscher

\[
\Delta E_n = \frac{4\pi a}{M L^3} n C_2 \left\{ 1 - \left( \frac{\bar{a}}{\pi L} \right) I + \left( \frac{\bar{a}}{\pi L} \right)^2 \left[ I^2 + (2n - 5) J \right] - \left( \frac{\bar{a}}{\pi L} \right)^3 \left[ I^3 + (2n - 7) I J + (5n^2 - 41n + 63) K \right] \right\} + n C_3 \frac{1}{L^6} \hat{\eta}_3^L + O(L^{-7})
\]

[Bogoliubov '47][Huang,Yang '57][Beane, WD, Savage PRD76;074507, 2007; WD+Savage PRD77:057502,2008]
Many mesons in LQCD

• Consider $\pi^+$ correlator ($m_u = m_d$)

\[
C_{-+}(t) = \left\langle 0 \left| \left( \sum_x \bar{d}(x,t) \gamma_5 \bar{u}(0,0) \right) \right| 0 \right\rangle \\
\rightarrow A e^{-E t}
\]
Many mesons in LQCD

- Consider \( n \pi^+ \) correlator (\( m_u=m_d \))

\[
C_n(t) = \left\langle 0 \left| \left( \sum_x \bar{d} \gamma_5 u(x, t) \bar{u} \gamma_5 d(0, 0) \right)^n \right| 0 \right\rangle \\
\to A e^{-E_n t}
\]
Many mesons in LQCD

- Consider $n \pi^+$ correlator ($m_u=m_d$)

$$C_n(t) = \left\langle 0 \left| \left[ \sum_x \bar{d} \gamma_5 u(x, t) \bar{u} \gamma_5 d(0, 0) \right]^n \right| 0 \right\rangle$$

$$\rightarrow A e^{-E_n t}$$

- $n!^2$ Wick contractions: $(12!)^2 \sim 10^{17}$

$$C_3(t) = \text{tr} [\Pi]^3 - 3 \text{tr} [\Pi] \text{tr} [\Pi^2 + 2 \text{tr} [\Pi^3]$$

$$\Pi = \sum_x \gamma_5 S(x, t; 0) \gamma_5 S^\dagger(x, t; 0)$$

- Maximal isospin: only a single quark propagator
Lattice details

- Calculations use MILC gauge configurations
  - \( L = 2.5 \text{ fm}, \ a = 0.12 \text{ fm}, \text{rooted} \ \text{staggered} \)
  - also \( L = 3.5 \text{ fm} \) and \( a = 0.09 \text{ fm} \)

- NPLQCD: domain-wall quark propagators
  - \( m_\pi \sim 291, \ 318, \ 352, \ 358, \ 491, \ 591 \ \text{MeV} \)
  - 24 propagators / lattice in best case
  - \( l_z = n = 1, \ldots, 12 \) pion and \( (S = n) \) kaon systems
n-meson energies

- Effective energy plots: $\log[C_n(t)/C_n(t+1)]$
Bosons in a box

- Large volume expansion of GS energy of n meson system to \(1/L^7\)

- 2 & 3 body interactions (N body: \(L^{-3(N-1)}\))

- \(n=2\): reproduces expansion of Lüscher

\[
\Delta E_n = \frac{4\pi a}{M L^3} n C_2 \left\{ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 \left[ I^2 + (2n - 5) J \right] - \left( \frac{a}{\pi L} \right)^3 \left[ I^3 + (2n - 7) I J + (5n^2 - 41n + 63) K \right] \right\} + n C_3 \frac{1}{L^6} \hat{n}_3^L + \mathcal{O}(L^{-7})
\]

[Beane, WD, Savage PRD76;074507, 2007; WD+Savage PRD77:057502, 2008]
Pion scattering

![Graph showing pion scattering with labels for LO, NLO, NNLO, and N^3LO orders, along with error bars for different n values (m=3,...,8).]
\[ \pi^+ \pi^+ \pi^+ \text{ interaction} \]

\[ m_{\pi} \hat{f}_{\pi}^{A_{\eta_3}} (L=2.5 \text{ fm}) \]

- \[ n=2 \text{ & } 7 \]

\[ m_\pi = 352 \text{ MeV} \]
$2\pi^+$ and $2K^-$ interaction

- Scattering lengths

curves: Weinberg
3π⁺ and 3K⁻ interaction

- First QCD three body interaction

Naïve dimension analysis: 1
Equation of State

- For large $n$: Bose-Einstein condensate
- $1/L$ expansion: analytic form of EOS
- Chemical potential $\mu(\rho)$ numerically using finite difference

\[
\mu = \frac{d E}{d n} \bigg|_{V \text{ const}}
\]

- Compare with LO $\chi$PT [Son & Stephanov]
Kaon Chemical Potential

\( \mu_{K^-} / m_K - 1 \)

\( \mu_{K^-} / m_K \)

\( \rho_{K^-} \) (2.5 fm)³

\( m_\pi \approx 291 \text{ MeV} \)
\( m_K \approx 580 \text{ MeV} \)

\( m_\pi \approx 352 \text{ MeV} \)
\( m_K \approx 597 \text{ MeV} \)

\( m_\pi \approx 491 \text{ MeV} \)
\( m_K \approx 640 \text{ MeV} \)

\( m_\pi \approx 591 \text{ MeV} \)
\( m_K \approx 678 \text{ MeV} \)

- 2+3 body fit
- No 3 body
- LOχPT
Color screening

[WD+ M Savage PRL 09]

- Static quark potential

\[ \rightarrow \#e^{-[V(R)]\#\delta V(R,n)]t} \]

- Modified by condensate? Hadronic screening?
In medium effects

\[ G_{n,W}(R, t_\pi, t_W, t) = \frac{C_{n,W}(R, t_\pi, t_W, t)}{C_n(t_\pi, t)C_W(R, t_w, t)} \rightarrow \# \exp \left[ -\delta V(R, n)(t - t_w) \right] \]
\( \delta V(R, n=1 \& 5) \)

DWF on MILC: \( a=0.09 \text{ fm}, 28^3 \times 96, m_\pi=318 \text{ MeV} \)
Pion screening

- \( r \) independent reduction in \( \bar{Q}Q \) force
- Dielectric medium inside flux tube
- Small effect: \( \delta F(n=1)/F = 0.002 \) at large \( R \)
- Hadronic medium effect
- Relevance to \( J/\psi \) suppression @ SPS/RHIC?
Few baryon systems

NPLQCD Phys. Rev D80, 074501, 2009
The problem with baryons ...

- QCD functional integrals done by importance sampling: propagators

- Variance in correlator determined by

\[ \sigma^2(C) = \langle CC^\dagger \rangle - |\langle C \rangle|^2 \]

- For nucleon:

\[
\frac{\text{signal}}{\text{noise}} \sim \exp \left[ - \left( M_N - 3/2m_\pi \right) t \right]
\]

- For nucleus A:

\[
\frac{\text{signal}}{\text{noise}} \sim \exp \left[ - A \left( M_N - 3/2m_\pi \right) t \right]
\]

[Lepage '89]
$\Xi^0\Xi^0 n$: effective energy

Phys. Rev D80, 074501, 2009

[Matrix-Prony analysis]
Golden window

- range of time-slices where s/n is const
- rearrangement cost in noise correlator
- heuristic scaling

\[ t_{\text{noise}} \sim \frac{1}{M_n - 3/2m_\pi} \ln \left[ \frac{m_\pi^3 L^3}{A^2} \right] \]

- 4, 5, ... baryon systems possible at current statistics!
Few baryon systems

- NPLQCD: $m_{\pi}=390$ MeV
- triton
- $\Xi^0\Xi^0 n$
- Yamazaki, et al [PACS-CS]: quenched $m_{\pi}=800$ MeV
  - $^3$He
  - $^4$He

Yamazaki et al. PRD 2010
Complex systems

[WD, Savage, Phys. Rev. D82, 014501, 2010]
Large systems

- How do we deal with complexity of contractions?
- One species: $N_{\text{terms}} \sim e^{\pi \sqrt{\frac{2n}{3}} / \sqrt{n}}$ [Ramanujan & Hardy]
- Two-species is harder, more is unfeasible
- How do we go beyond $n=12$?
  - Previous method fails because of Pauli principle
  - Avoid by using multiple propagator sources but this leads to contraction complexity
Few pion contractions

\[ C_{1\pi}(t) = \]

\[ C_{2\pi}(t) = \]

\[ C_{3\pi}(t) = \]
Blocks

- Define a partly contracted pion correlator

\[ \Pi \equiv R_1 = \sum_x S_u(x, t; x_0) \gamma_5 S_d(x_0, x; t) \gamma_5 = \sum_x S_u(x, t; x_0) S_d^\dagger(x, t; x_0) \]

- Time-dependent 12x12 matrix (spin-colour indices)

- Correlators

\[ C_1(t) = \langle \Pi \rangle, \quad C_2(t) = \langle \Pi \rangle^2 - \langle \Pi^2 \rangle, \ldots \]

- Functional definition

\[ \Pi_{ij} = \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_1(t) \]

- Generalises to

\[ (R_n)_{ij} \equiv \bar{u}_i(x) u_k(x_0) \frac{\delta}{\delta \bar{u}_j(x) \delta u_k(x_0)} C_n(t) \]
Recursion relation

[WD, Savage, Phys. Rev. D82, 014501, 2010]

• The block objects are simply related
• Recursion relation

\[ R_{n+1} = \langle R_n \rangle R_1 - n \ R_n \ R_1 \]
• Initial condition is that \( R_1 = \Pi, \quad R_j = 0, \ \forall j < 1 \)
• Can also construct a descending recursion as we know that \( R_{13}=0 \)
Multi-source systems

- To get beyond n=12, need to consider multi-source systems

- Consider two sources first

\[
C_{(n_1 \pi_1^+, \ n_2 \pi_2^+)}(t) = \left\langle \left( \sum_x \pi^+(x, t) \right)^{n_1+n_2} \left( \pi^-(y_1, 0) \right)^{n_1} \left( \pi^-(y_2, 0) \right)^{n_2} \right\rangle
\]

- \( C_{(2,1)}(t) \) contains contractions like

\[
\begin{array}{c}
\text{y}_1 \\
\text{x, t} \\
\text{y}_2
\end{array}
\]
Multi-source systems

- Multiple types of blocks needed

\[ A_{ab} = \sum_x S_u(x, t; x_a) S_d^\dagger(x, t; x_b) \]

- Two species case has a simple recursion relation:
  First define

\[ P_1 = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \]

Then the generalisations of the \( R_n \) satisfy

\[ Q_{(n_1+1,n_2)} = \langle Q_{(n_1,n_2)} \rangle P_1 - (n_1 + n_2) Q_{(n_1,n_2)} P_1 \]

\[ + \langle Q_{(n_1+1,n_2-1)} \rangle P_2 - (n_1 + n_2) Q_{(n_1+1,n_2-1)} P_2 \]
Extensions

• Recursions also constructed for
  • $m$-source systems
  • $k$-species systems: $\pi$'s, K’s, D’s, B’s, ...
  • $m$-source, $k$-species systems

$$T_{n+1_{rs}} = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle T_{n+1_{rs}-1_{ij}} \rangle P_{ij} - \bar{N} T_{n+1_{rs}-1_{ij}} P_{ij}$$

where subscripts are matrices

• Implemented as matrix multiplications - computationally tractable
Extensions

- Factorial cost reduced
- Each iteration involves essentially two-body contractions
- Codes running for up to \( \sim 100 \) mesons
- Recursions also exist for baryon contractions but are messier
- Different choices of basis objects for recursion
- Should allow calculations of \( B=4,5,\ldots \) systems
Summary

• LQCD is making progress in many-body systems
• Explore novel types of QCD matter
• Properties and effects of meson condensates
• New algorithms to ameliorate contraction complexity
• (Light) Nuclei?