58.) $\hat{A}\hat{B} = \begin{pmatrix} -1 & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{3} & -2 \end{pmatrix} \begin{pmatrix} 10 & \sqrt{6} & -\sqrt{2} \\ \sqrt{6} & 9 & \sqrt{3} \\ -\sqrt{2} & \sqrt{3} & 11 \end{pmatrix}$

$$= \begin{pmatrix} -10+6-2 & -\sqrt{6}+9\sqrt{6}+\sqrt{6} & \sqrt{2}+3\sqrt{2}+11\sqrt{2} \\ 10\sqrt{6}-\sqrt{6} & 6+3 & -2\sqrt{3}+11\sqrt{3} \\ 10\sqrt{2}+3\sqrt{2}+2\sqrt{2} & 2\sqrt{3}+9\sqrt{3}-2\sqrt{3} & -2+3-22 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & +9\sqrt{6} & +15\sqrt{2} \\ 9\sqrt{6} & 9 & +9\sqrt{3} \\ 15\sqrt{2} & 9\sqrt{3} & -21 \end{pmatrix}$$

$\underline{\underline{\hat{B}\hat{A}}} = \begin{pmatrix} 10 & \sqrt{6} & -\sqrt{2} \\ \sqrt{6} & 9 & \sqrt{3} \\ -\sqrt{2} & \sqrt{3} & 11 \end{pmatrix} \begin{pmatrix} -1 & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{3} & -2 \end{pmatrix}$

$$= \begin{pmatrix} -10+6-2 & 10\sqrt{6}-\sqrt{6} & 10\sqrt{2}+3\sqrt{2}+2\sqrt{2} \\ -\sqrt{6}+9\sqrt{6}+\sqrt{6} & 6+3 & 2\sqrt{3}+9\sqrt{3}-2\sqrt{3} \\ \sqrt{2}+3\sqrt{2}+11\sqrt{2} & -2\sqrt{3}+11\sqrt{3} & -2+3-22 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 9\sqrt{6} & 15\sqrt{2} \\ 9\sqrt{6} & 9 & 9\sqrt{3} \\ 15\sqrt{2} & 9\sqrt{3} & -21 \end{pmatrix} = \underline{\underline{\hat{A}\hat{B}}}$$

<u>Eigenwerte von $\hat{A}$:</u>

$\det(\hat{A} - \lambda\hat{1}) = \begin{vmatrix} -1-\lambda & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & -\lambda & \sqrt{3} \\ \sqrt{2} & \sqrt{3} & -2-\lambda \end{vmatrix} = -\lambda(1+\lambda)(2+\lambda) + 6 + 6 + 2\lambda + 3(1+\lambda) + 6(2+\lambda)$

$$= -\cancel{2\lambda} - 3\lambda^2 - \lambda^3 + 12 + \cancel{2\lambda} + 3 + 3\lambda + 12 + 6\lambda$$

$$= -\lambda^3 - 3\lambda^2 + 9\lambda + 27$$

$$= -(\lambda-3)(+\lambda^2+6\lambda+9) = -(\lambda-3)(\lambda+3)^2$$

$\Rightarrow$ $\underline{\underline{\lambda_1 = 3, \quad \lambda_2 = \lambda_3 = -3}}$

<u>EV zu $\lambda_1 = 3$:</u> $\begin{pmatrix} -4 & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & -3 & \sqrt{3} \\ \sqrt{2} & \sqrt{3} & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$$\Rightarrow \quad -4x + \sqrt{6}\,y + \sqrt{2}\,z = 0$$
$$\sqrt{6}\,x - 3y + \sqrt{3}\,z = 0$$

$$x = 1 \Rightarrow \quad \sqrt{6}\,y + \sqrt{2}\,z = 4 \quad / \sqrt{3}$$
$$-3y + \sqrt{3}\,z = -\sqrt{6} \quad / -\sqrt{2}$$

$$3\sqrt{2}\,y + \sqrt{6}\,z = 4\sqrt{3}$$
$$3\sqrt{2}\,y - \sqrt{6}\,z = 2\sqrt{3}$$

$$y = \frac{6\sqrt{3}}{6\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$z = (4\sqrt{3} - 3\sqrt{3}) / \sqrt{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \underline{\underline{\vec{v}_1}} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{3/2} \\ 1/\sqrt{2} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix}}}$$

$\vec{v}_1$ ist EV von $\hat{B}$:

$$\underline{\underline{\hat{B}\,\vec{v}_1}} = \begin{pmatrix} 10 & \sqrt{6} & -\sqrt{2} \\ \sqrt{6} & 9 & \sqrt{3} \\ -\sqrt{2} & \sqrt{3} & 11 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 10/\sqrt{3} + \sqrt{3} - 1/\sqrt{3} \\ \sqrt{2} + 9/\sqrt{2} + 1/\sqrt{2} \\ -\sqrt{2}/\sqrt{3} + \sqrt{3}/\sqrt{2} + 11/\sqrt{6} \end{pmatrix}$$

$$= 12\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} = \underline{\underline{12\,\vec{v}_1}} \qquad \underline{\underline{\tilde{\lambda}_1 = 12}}$$

EVn zu $\lambda_2 = \lambda_3 = -3$ nicht eindeutig
$\Rightarrow$ müssen EVn mittels $\hat{B}$ bestimmen

Eigenwerte von $\hat{B}$:

$$\det(\hat{B} - \lambda\hat{1}) = \begin{vmatrix} 10-\lambda & \sqrt{6} & -\sqrt{2} \\ \sqrt{6} & 9-\lambda & \sqrt{3} \\ -\sqrt{2} & \sqrt{3} & 11-\lambda \end{vmatrix} =$$

$$= (10-\lambda)(9-\lambda)(11-\lambda) - 6 - 6 - 2(9-\lambda)$$
$$-3(10-\lambda) - 6(11-\lambda) =$$
$$= -\lambda^3 + 30\lambda^2 - 288\lambda + 864 = -(\lambda-12)^2(\lambda-6)$$

$$\Rightarrow \quad \underline{\underline{\tilde{\lambda}_1 = \tilde{\lambda}_2 = 12 \quad \tilde{\lambda}_3 = 6}}$$

EV zu $\tilde{\lambda}_3 = 6$:
$$\begin{pmatrix} 4 & \sqrt{6} & -\sqrt{2} \\ \sqrt{6} & 3 & \sqrt{3} \\ -\sqrt{2} & \sqrt{3} & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \quad 4x + \sqrt{6}\,y - \sqrt{2}\,z = 0$$
$$\sqrt{6}\,x + 3y + \sqrt{3}\,z = 0$$

$$x = 1 \Rightarrow \quad \sqrt{6}\,y - \sqrt{2}\,z = -4 \quad | \sqrt{3}$$
$$3y + \sqrt{3}\,z = -\sqrt{6} \quad | \sqrt{2}$$

$$3\sqrt{2}\,y - \sqrt{6}\,z = -4\sqrt{3}$$
$$3\sqrt{2}\,y + \sqrt{6}\,z = -2\sqrt{3}$$

$$y = -\sqrt{\frac{3}{2}}$$

$$z = (-2\sqrt{3} + 3\sqrt{3}) / \sqrt{6} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \underline{\underline{\vec{v}_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix}}}$$

Da $\hat{A}$ und $\hat{B}$ hermitesche Matrizen $\Rightarrow$ EVn bilden ONS

$\Rightarrow \vec{v}_2 \perp \vec{v}_1, \vec{v}_3$

$$\underline{\underline{\vec{v}_2}} = \vec{v}_3 \times \vec{v}_1 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1/\sqrt{3} \\ 0 \\ \sqrt{2/3} \end{pmatrix}}}$$

Gemeinsames System von EVn

$$\underline{\underline{\vec{v}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1/\sqrt{3} \\ 0 \\ \sqrt{2/3} \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix}}}$$

~~11~~, 59, a) $\tilde{f}_1(x) = x \,, \quad \tilde{f}_2(x) = x^3$

$$\|\tilde{f}_1\|^2 = (\tilde{f}_1, \tilde{f}_1) = \int_0^1 dx\, x^2 = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\Rightarrow f_1 = \frac{\tilde{f}_1}{\|\tilde{f}_1\|} = \sqrt{3}\, x$$

$$\hat{f}_2 = \tilde{f}_2 - f_1 (f_1, \tilde{f}_2) = x^3 - \sqrt{3}\, x \int_0^1 dx' (\sqrt{3}\, x') x'^3$$

$$= x^3 - 3x \frac{x'^5}{5}\Big|_0^1 = x^3 - \frac{3}{5}x$$

$\Rightarrow$ Orthogonalbasis $\{x, \, x^3 - \frac{3}{5}x\}$

b) Allgemeines $f(x) = \alpha x + \beta x^3 = \alpha x + \beta(x^3 - \frac{3}{5}x) + \frac{3}{5}\beta x$

$$= (\alpha + \tfrac{3}{5}\beta)x + \beta(x^3 - \tfrac{3}{5}x)$$

$$x \frac{d}{dx} f(x) = x \frac{d}{dx}(\alpha x + \beta x^3) = x(\alpha + 3\beta x^2)$$

$$= \alpha x + 3\beta x^3 = \alpha x + 3\beta(x^3 - \tfrac{3}{5}x) + \tfrac{9}{5}\beta x$$

$$= (\alpha + \tfrac{9}{5}\beta)x + 3\beta(x^3 - \tfrac{3}{5}x)$$

<u>Komponentendarstellung:</u>

	alte Basis	neue Basis
f	$\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$	$\begin{pmatrix}\alpha + \frac{3}{5}\beta \\ \beta\end{pmatrix}$
Af	$\begin{pmatrix}\alpha \\ 3\beta\end{pmatrix}$	$\begin{pmatrix}\alpha + \frac{9}{5}\beta \\ 3\beta\end{pmatrix}$
A	$\begin{pmatrix}1 & 0 \\ 0 & 3\end{pmatrix}$	$\begin{pmatrix}1 & \frac{6}{5} \\ 0 & 3\end{pmatrix}$

c) $$\begin{pmatrix}\alpha + \frac{3}{5}\beta \\ \beta\end{pmatrix} = \hat{S}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \quad \Rightarrow \quad \underline{\underline{\hat{S} = \begin{pmatrix}1 & \frac{3}{5} \\ 0 & 1\end{pmatrix}}}$$