Integration measure over the group, or over the group parameters (angles)

Invariant integration measure (called **Haar measure**) on a group is such that a change of variables (angles) caused by a group multiplication, does not change the measure.

For example, in flat space the measure dV = dx dy dz is invariant under the group of translations, since d(x+a) d(y+b) d(z+c) = dx dy dz.

One has do find an analogous mesure in terms of angles, such that

$$\mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\, \mathrm{f}(\alpha,\beta,\gamma) = \mathrm{d}\alpha'\, \mathrm{d}\beta'\, \mathrm{d}\gamma'\, \mathrm{f}(\alpha',\beta',\gamma')$$

where α,β,γ is the parameterization of *U,* and α', β', γ' is the parameterization of *U' = VU* where *V* is an arbitrary element of the group, or *U' = UV*, or both.

In short, the left- and right-invariant Haar measure is such that

$$\int dU = \int d(UV) = \int d(VU) \qquad \left[\text{similar to} \quad \int dx = \int d(x + \mathrm{const})\right]$$

$$\int dU \left(U^\dagger\right)^\alpha_\beta U^\gamma_\delta = \frac{1}{N}\delta^\alpha_\delta\,\delta^\gamma_\beta$$

for SU(N)

$$\int dO\, O^{ab}\, O^{cd} = \frac{2}{(N-1)N}\delta^{ac}\,\delta^{bd}$$

for SO(N)

General method of constructing invariant measures

Let g,h,k belong to some Lie group G. Then g’ = h g and g” = g k also are elements of G. Construct the left- and right- invariant one-forms:

$$L_m = i g^{-1}\cdot\frac{\partial g}{\partial\alpha_m} = i\,(h\cdot g)^{-1}\cdot\frac{\partial(h\cdot g)}{\partial\alpha_m} = i g'^{-1}\cdot\frac{\partial g'}{\partial\alpha_m} = i g'^{-1}\cdot\frac{\partial g'}{\partial\alpha'_p}\frac{\partial\alpha'_p}{\partial\alpha_m} = L'_p\frac{\partial\alpha'_p}{\partial\alpha_m}$$

$$R_m = i\frac{\partial g}{\partial\alpha_m}\cdot g^{-1} = i\frac{\partial(g\cdot k)}{\partial\alpha_m}\cdot(g\cdot k)^{-1} = i\frac{\partial g''}{\partial\alpha_m}g''^{-1} = i\frac{\partial g''}{\partial\alpha''_p}g''^{-1}\frac{\partial\alpha''_p}{\partial\alpha_m} = R''_p\frac{\partial\alpha''_p}{\partial\alpha_m}$$

where α’s are d angles parameterizing the group in question, d is the dimension of the group, or dimension of the space of group parameters. Now construct a d x d “metric tensor”

$$g_{mn} = 2\mathrm{Tr}(L_m L_n) = 2\mathrm{Tr}(R_m R_n) = 2\mathrm{Tr}(L'_p L'_q)\frac{\partial\alpha'_p}{\partial\alpha_m}\frac{\partial\alpha'_q}{\partial\alpha_n} = g'_{pq}\frac{\partial\alpha'_p}{\partial\alpha_m}\frac{\partial\alpha'_q}{\partial\alpha_n}$$

$$\det\,(\mathrm{g}_{\mathrm{mn}}) = \det\,(\mathrm{g}'_{\mathrm{pq}})\times\left(\text{Jacobian}\left(\frac{\partial\alpha'}{\partial\alpha}\right)\right)^2$$

Therefore, the left- *and* right-invariant integration measure is

$$dU = d\alpha_1 \ldots d\alpha_d \sqrt{\det(g_{mn})} = d\alpha_1' \ldots d\alpha_d' \sqrt{\det(g'_{mn})} = d(VU) = d(UV)$$

since

$$d\alpha_1 \ldots d\alpha_d \ \text{Jacobian}\left(\frac{\partial \alpha'}{\partial \alpha}\right) = d\alpha_1' \ldots d\alpha_d'$$

Let us consider a couple of examples.

$SU(2)$, 3 parameters

$$U_2 = \begin{pmatrix} e^{-i\alpha_{11}} \cos\phi_1 & e^{i\alpha_{12}} \sin\phi_1 \\ -e^{-i\alpha_{12}} \sin\phi_1 & e^{i\alpha_{11}} \cos\phi_1 \end{pmatrix}$$

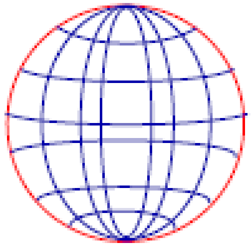
where the last column can be viewed as a $2d$ complex vector $v_2 = (z^1, z^2)$ normalized as $|z^1|^2 + |z^2|^2 = 1$, which defines an S^3 sphere. The first column is the orthogonal vector $v_1^i = \epsilon^{ij} \bar{v}_{2j}$.

The group measure can be written as an integral over the S^3 sphere,

$$\frac{1}{\pi^2}\int dz^1 d\bar{z}^1 dz^2 d\bar{z}^2\, \delta(|z^1|^2 + |z^2|^2 - 1),$$

or, explicitly in terms of three angles, as

$$\frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}} d\phi_1 \sin\phi_1 \cos\phi_1 \int_0^{2\pi} d\alpha_{11} \int_0^{2\pi} d\alpha_{12} \qquad (=1)$$



3-sphere $S^3 : \phi, \alpha_1, \alpha_2$

Exercise. Compute explicitly the metric tensor $g_{mn} = \mathrm{Tr}\left(iU^\dagger \cdot \frac{\partial U}{\partial \alpha_m} \cdot iU^\dagger \cdot \frac{\partial U}{\partial \alpha_n} \right) = \mathrm{Tr}\left(\frac{\partial U^\dagger}{\partial \alpha_m} \cdot \frac{\partial U}{\partial \alpha_n} \right)$

for this particular parameterization of U and show that in this particular case

$$\sqrt{\det\,(g_{mn})} = \sin\phi_1 \cos\phi_1$$

$SU(3)$, 8 parameters

We build the parametrization iteratively:

$$R_2 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{U_2}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array} \right),$$

where U2 is a general parameterization of SU(2)

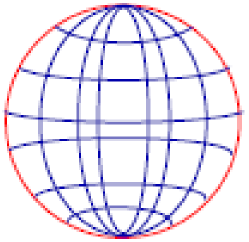
and define (S3 is a special SU(3) matrix, with only 5 parameters)

$$U_3 = S_3 R_2, \qquad S_3 = \begin{pmatrix} e^{i\alpha_{23}} \cos\theta & 0 & e^{i\alpha_{23}} \sin\theta \\ -e^{i\alpha_{22}} \sin\theta \sin\phi_2 & e^{-i\alpha_{21}-i\alpha_{23}} \cos\phi_2 & e^{i\alpha_{22}} \cos\theta \sin\phi_2 \\ -e^{i\alpha_{21}} \sin\theta \cos\phi_2 & -e^{-i\alpha_{22}-i\alpha_{23}} \sin\phi_2 & e^{i\alpha_{21}} \cos\theta \cos\phi_2 \end{pmatrix}$$

The last column can be viewed as a $3d$ complex vector $v_3 = (z^1, z^2, z^3)$ normalized to $|z^1|^2 + |z^2|^2 + |z^3|^2 = 1$, which defines an S^5 sphere. The three columns are constructed as (complexified) orts in spherical coordinates: $v_1 \sim e_r$, $v_2 \sim e_\phi$, $v_3 \sim e_\theta$. We use part of the freedom of choosing the orts and the angles in such a way that $U_3 = 1_3$ when all angles are set to zero.

The measure on S^5 can be written as

$$\frac{2}{\pi^3}\int dz^1 d\bar{z}^1 dz^2 d\bar{z}^2 dz^3 d\bar{z}^3\, \delta(|z^1|^2+|z^2|^2+|z^3|^2-1)$$



5-sphere S^5 : $\theta,\phi,\alpha_1,\alpha_2,\alpha_3$

or, explicitly in terms of five angles, as

$$\frac{1}{\pi^3}\int_0^{\frac{\pi}{2}} d\theta\ \cos^3\theta \sin\theta \int_0^{\frac{\pi}{2}} d\phi_2\ \sin\phi_2 \cos\phi_2 \int_0^{2\pi} d\alpha_{21} \int_0^{2\pi} d\alpha_{22} \int_0^{2\pi} d\alpha_{23} \qquad (=1).$$

The integrations limits are chosen such that the S^5 sphere is covered once.

The full $SU(3)$ measure is found in the standard way: one constructs the metric tensor

$$g_{mn} = \mathrm{Tr}\,\frac{\partial U_3}{\partial\beta^m}\frac{\partial U_3^\dagger}{\partial\beta^n}, \qquad \beta^m = \alpha_{11}, \alpha_{12}, \phi_1, \alpha_{21}, \alpha_{22}, \alpha_{23}, \phi_2, \theta, \quad m, n = 1\ldots 8;$$

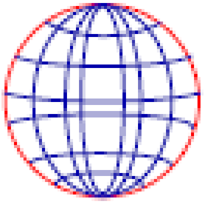
then the $SU(3)$ measure is

$$\sqrt{\det g} \sim (\sin\phi_1 \cos\phi_1)\cdot(\cos^3\theta \sin\theta \sin\phi_2 \cos\phi_2)$$

i.e. it is **factorized** into the product of the measures over the spheres S^3 and S^5 !

Conclusion

A general $SU(3)$ matrix can be written through 8 “Euler angles” parameterizing the $S^3 \times S^5$ spheres



3-sphere $S^3: \phi_1, \alpha_{11}, \alpha_{12}$

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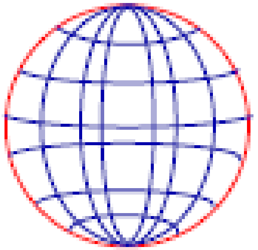

5-sphere $S^5: \theta, \phi_2, \alpha_{21}, \alpha_{22}, \alpha_{23}$

In general, $SU(N)$ can be parameterized by N^2-1 "Euler angles" on the odd-dimensional spheres: $S^3 \times S^5 \times \ldots \times S^{2N-1}$.

$$\sum_{n=2}^{N}(2n-1) = N^2 - 1$$

The manifold of parameters of SU(N) is a direct product of odd-dimensional spheres,

and the invariant integration measure is a product of measures for those spheres !

(see the proof on next page)

Proof of the factorization of the integrations measure over SU(N)

Let us start from SU(3): $U_3(\alpha_1,\ldots,\alpha_8) = S_3(\alpha_4,\ldots,\alpha_8)\cdot U_2(\alpha_1,\alpha_2,\alpha_3)$

The `left’ one-form $L_m = -iU_3^\dagger \partial_m U_3 = -iU_2^\dagger \cdot S_3^\dagger \partial_m S_3\, U_2 - iU_2^\dagger\, \partial_m U_2\ , \qquad \partial_m \equiv \dfrac{\partial}{\partial \alpha_m}$

Introduce the `left vielbein’ $\mathrm{e}_m^a = 2\mathrm{Tr}\left(L_m t^a\right) = -\,2i\mathrm{Tr}(U_2^\dagger \partial_m U_2 t^a) - 2\,i\mathrm{Tr}(S_3^\dagger \partial_m S_3)(U_2 t^a U_2^\dagger)$

t^a are SU(3) generators:

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The first term in e_m^a has only m=1,2,3 and a=1,2,3 components;
The second term has m=4,5,6,7,8 and a=1,…,8 components. Therefore the 8-bein matrix is of the form

$$\begin{array}{cc} & \longrightarrow a \\ m\downarrow & \\ e_m^a = & \left(\begin{array}{c|c} e_{1,2,3}^{1,2,3} & 0 \\ \hline e_{1,2,3}^{4,5,6,7,8} & e_{4,5,6,7,8}^{4,5,6,7,8} \end{array}\right) \end{array}$$

The metric tensor of the group parameter space

$$g_{mn} = 2\mathrm{Tr}\left(L_m L_n\right) = 4\mathrm{Tr}\left(L_m t^a\right)\mathrm{Tr}\left(L_n t^a\right) = e_m^a e_n^a, \qquad \det g_{mn} = \left(\det \boldsymbol{e}_{\boldsymbol{m}}^{\boldsymbol{a}}\right)^2$$

$$\sqrt{\det g} = \det e = \det\left(\tilde{e}_{4,5,6,7,8}^{4,5,6,7,8}\right)\det\left(e_{1,2,3}^{1,2,3}\right) \quad = \sqrt{\det g(S^5)}\sqrt{\det g(S^3)}$$

$$\tilde{e}_m^a = -2i\mathrm{Tr}(S_3^\dagger \partial_m S_3)(U_2 t^a U_2^\dagger) = -2i\mathrm{Tr}(S_3^\dagger \partial_m S_3 \tilde{t}^a), \qquad \tilde{t}^a = U_2 t^a U_2^\dagger .$$

Similarly, for any SU(N),

$$\sqrt{\det g(SU(N))} = \sqrt{\det g(S^{2N-1})}\sqrt{\det g(SU(N-1))}$$

$$\sqrt{\det g(SU(N))} = \sqrt{\det g(S^{2N-1})}\ldots\sqrt{\det g(S^3)}$$

Integration measure over SU(N) is a product of integrations measures over odd spheres!

Explicit parameterization of SO(N)

An alternative (Euler angles) parameterization of SU(2):

$$U = \exp\left(i\alpha\frac{\sigma^3}{2}\right)\cdot\exp\left(i\beta\frac{\sigma^2}{2}\right)\cdot\exp\left(i\gamma\frac{\sigma^3}{2}\right) \quad = \begin{pmatrix} e^{\frac{1}{2}i(\alpha+\gamma)}\cos\left(\frac{\beta}{2}\right) & e^{\frac{1}{2}i(\alpha-\gamma)}\sin\left(\frac{\beta}{2}\right) \\ -e^{-\frac{1}{2}i(\alpha-\gamma)}\sin\left(\frac{\beta}{2}\right) & e^{-\frac{1}{2}i(\alpha+\gamma)}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}$$

From a general 2 x 2 SU(2) matrix one can build a general 3 x 3 SO(3) matrix according to the rule:

$$\mathrm{O}^{ab} = \frac{1}{2}\mathrm{Tr}\left(U^\dagger\sigma^a U\sigma^b\right) =$$

$$\begin{pmatrix} \cos(\alpha)\cos(\beta)\cos(\gamma)-\sin(\alpha)\sin(\gamma) & \cos(\gamma)\sin(\alpha)+\cos(\alpha)\cos(\beta)\sin(\gamma) & -\cos(\alpha)\sin(\beta) \\ -\cos(\beta)\cos(\gamma)\sin(\alpha)-\cos(\alpha)\sin(\gamma) & \cos(\alpha)\cos(\gamma)-\cos(\beta)\sin(\alpha)\sin(\gamma) & \sin(\alpha)\sin(\beta) \\ \cos(\gamma)\sin(\beta) & \sin(\beta)\sin(\gamma) & \cos(\beta) \end{pmatrix}$$

The metric tensor of the group space

$$g_{mn} = 2\mathrm{Tr}\left(\partial_m U^\dagger \partial_n U\right) = \frac{1}{2}\mathrm{Tr}\left(\partial_m O^T \partial_n O\right) = \begin{pmatrix} 1 & 0 & \cos(\beta) \\ 0 & 1 & 0 \\ \cos(\beta) & 0 & 1 \end{pmatrix}$$

$$\sqrt{\det g_{mn}} = \sin\beta.$$

As in SU(N), a parameterization of SO(N) can be built iteratively: $O_N = S_N O_{N-1}$

where O_{N-1} is a general SO(N-1) matrix put in the left upper corner of the N x N matrix, and S_N is a SO(N) matrix of a special type, built of orths, *e.g.*

$$S_4 = \begin{pmatrix} \cos(\chi) & 0 & 0 & -\sin(\chi) \\ \sin(\theta)\sin(\chi) & \cos(\theta) & 0 & \cos(\chi)\sin(\theta) \\ \cos(\theta)\sin(\phi)\sin(\chi) & -\sin(\theta)\sin(\phi) & \cos(\phi) & \cos(\theta)\cos(\chi)\sin(\phi) \\ \cos(\theta)\cos(\phi)\sin(\chi) & -\cos(\phi)\sin(\theta) & -\sin(\phi) & \cos(\theta)\cos(\phi)\cos(\chi) \end{pmatrix}$$

Its metric is that of a 3-dim sphere (in general, (N-1)-dim sphere). Therefore, the space of parameters of SO(N) is a direct product of spheres:

$$SO(N) \quad \sim \quad S^{N-1} \times S^{N-2} \times \ldots \times S^3 \times S^3$$

$$SU(N) \quad \sim \quad S^{2N-1} \times S^{2N-3} \times \ldots \times S^3$$